Dropping Rational Expectations

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Dropping rational expectations

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Abstract

In [5], we proposed a general equilibrium model, with incomplete financial markets and asymmetric information, where agents forecasted prices privately without rational expectations. Consistently, they anticipated idiosyncratic sets of future prices, and elected probability laws on these sets, that we called beliefs. Under mild conditions, and differently from Hart [1975] and Radner [1979], equilibrium always existed in this model, as long as agents' anticipations precluded arbitrage. The joint determination of equilibrium prices and beliefs is traditionally seen as a rational expectations' problem. Hereafter, we suggest it may be otherwise. We propose to show that agents, whose prior anticipation sets yield an arbitrage, may update their expectations from observing trade opportunities on financial markets. With no price to be observed, they eventually infer smaller arbitrage-free anticipation sets, which cannot be narrowed down any further. Once these sets are attained, equilibrium prices may change if agents change their beliefs, but they will convey the same information.

Key words: anticipations, inferences, perfect foresight, existence problem, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

The traditional approach to sequential financial equilibrium relies on Radner's (1972-1979) classical, but restrictive, assumptions that agents have the so-called 'rational expectations' of private information signals, and 'perfect foresight' of future prices. These assumptions lead to well-known cases of inexistence of equilibrium, which followed Hart (1975) and Radner (1979). The joint determination of prices and beliefs is then explained by a price model from which agents may infer information, that is, agents are assumed to know how equilibrium prices are determined.

In [5], we dropped rational expectations and let agents form their forecasts and beliefs privately. The model we proposed led to opposite outcomes as the classical one: information and beliefs were typically asymmetric at equilibrium, whose full existence property could be restored, even in the cases of asymmetric information and real asset markets, studied, respectively, by Radner (1979) and Hart (1975). In our setting, agents, being unaware of other agents' forecasts (upon which equilibrium prices depend), need anticipate (idiosyncratic) sets of realizable prices in each state they expect, and elect probability laws on these sets, which we call '*beliefs*'.

In [5], we referred to '*anticipation structures*' as the collections of anticipation sets across agents, whose intersections were non-empty, and to '*structure of beliefs*' as the collections of beliefs, whose supports defined an anticipation structure. An arbitrage-free anticipation structure was one which granted no agent an unlimited arbitrage opportunity on financial markets. We also introduced a notion of '*minimum uncertainty set*', denoted by Δ, which embedded the incompressible uncertainty upon prices, stemming from private beliefs.

In recalling the model's results below, we implicitly assume that consumers are cautious enough to let their anticipation sets include Δ.

We showed in [5] that a sequential equilibrium existed (with prices clearing all current and future markets) if, and only if, agents' anticipation structure was arbitrage-free. When the anticipation structure was not arbitrage-free at the outset, agents, endowed with no price model, could yet narrow down their anticipation sets and infer an arbitrage-free anticipation structure, from observing a so called '*no-arbitrage price*' of assets. That inferred anticipation structure represented agents' ultimate information and could not be further refined, e.g., from observing subsequent equilibrium prices. In this sense, the joint determination of prices and beliefs at equilibrium escaped the rational expectation paradigm, since agents never used a price model. Yet, the appearance of a no-arbitrage price on markets remained unexplained. The main purpose of this paper is to address this issue formally, before possibly implementing the path to equilibrium into a strategic market game.

The no-arbitrage prices comprise all equilibrium asset prices. Yet, when the anticipation structure is not arbitrage-free at the outset, agents cannot agree on a price assessment of assets. We now propose to show that, in that case, agents may always narrow down their anticipation sets from observing mutually beneficial trade opportunities on financial markets. A trade-house might reveal these trades. That refinement process leads agents to infer the coarsest arbitrage-free anticipation structure refining the initial structure. It extends to this infinite dimensional setting the refinement path described in Cornet-De Boisdeffre (2009). Again, agents' refined anticipation sets, after being inferred without observing any price, cannot be narrowed down any further, even from observing equilibrium prices. The latter may change jointly with agents' beliefs, but convey no additional information. In this sense, the joint determination of beliefs and prices at equilibrium drops rational expectations. Indeed, agents use no price model in their path towards equilibrium.

The paper is organized as follows: in Section 2, we summarize the specification and properties of the model presented in [5]; in Section 3, we present the inference path towards equilibrium anticipations when no market price is available.

2 The basic model

Hereafter, we consider a pure-exchange economy with two periods ($t \in \{0, 1\}$), a commodity market and a financial market, where agents may be asymmetrically informed and form private price forecasts. The sets of agents, $I := \{1, ..., m\}$, commodities, $\mathcal{L} := \{1, ..., L\}$, states of nature, S, and assets, $\mathcal{J} := \{1, ..., J\}$, are all finite. Dropping proofs, we recall the main definitions, claims, and Theorem 1, of [5].

2.1 The model's notations

Throughout, we denote by $\cdot$ the scalar product and $\|.\|$ the Euclidean norm on an Euclidean space and by $\mathcal{B}(K)$ the Borel sigma-algebra of a topological set, K. We let $s = 0$ be the non-random state at $t = 0$ and $S' := \{0\} \cup S$. For all set $\Sigma \subset S'$ and tuple $(s, l, x, x', y, y') \in \Sigma \times \mathcal{L} \times \mathbb{R}^{\Sigma} \times \mathbb{R}^{\Sigma} \times \mathbb{R}^{L\Sigma} \times \mathbb{R}^{L\Sigma}$, we shall denote by:

- $x_s \in \mathbb{R}$, $y_s \in \mathbb{R}^L$ the scalar and vector, indexed by $s \in \Sigma$, of x, y, respectively;
- y_s^l the l^{th} component of $y_s \in \mathbb{R}^L$;
- $x \leqslant x'$ and $y \leqslant y'$ (respectively, $x << x'$ and $y << y'$) the relations $x_s \leqslant x'_s$ and $y_s^l \leqslant y_s'^l$ (resp., $x_s < x'_s$ and $y_s^l < y_s'^l$) for each $(l, s) \in \{1, ..., L\} \times \Sigma$;

- $x < x'$ (resp., $y < y'$) the joint relations $x \leqslant x'$, $x \neq x'$ (resp., $y \leqslant y'$, $y \neq y'$);

- $\mathbb{R}^{L\Sigma}_{+} = \{x \in \mathbb{R}^{L\Sigma} : x \geqslant 0\}$ and $\mathbb{R}^{\Sigma}_{+} := \{x \in \mathbb{R}^{\Sigma} : x \geqslant 0\}$,
  $\mathbb{R}^{L\Sigma}_{++} := \{x \in \mathbb{R}^{L\Sigma} : x >> 0\}$ and $\mathbb{R}^{\Sigma}_{++} := \{x \in \mathbb{R}^{\Sigma} : x >> 0\}$;

- $\mathcal{M}_0 := \{(p_0, q) \in \mathbb{R}^L_+ \times \mathbb{R}^J : \|p_0\| + \|q\| = 1\}$;

- $\mathcal{M}_s := \{(s, p) \in S \times \mathbb{R}^L_+ : \ \|p\| = 1\}$, for every $s \in S$;

- $\mathcal{M} := \cup_{s \in S} \mathcal{M}_s$, a topological subset of the Euclidean space $\mathbb{R}^{L+1}$;

- $B(\omega, \varepsilon) := \{\omega' \in \mathcal{M}\text{: } \|\omega' - \omega\| < \varepsilon\}$, for every pair $(\omega, \varepsilon) \in \mathcal{M} \times \mathbb{R}_{++}$;

- $P(\pi) := \{\omega \in \mathcal{M} : \ \pi(B(\omega, \varepsilon)) > 0, \forall \varepsilon > 0\}$, the support of a probability, π, on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$;

- $\pi(P)$, for any closed set, $P \subset \mathcal{M}$, the set of probabilities on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, whose support (as defined above) is P.

2.2 The commodity and asset markets

Consumption goods may be exchanged by consumers, on the spot markets of both periods. In each state, $s \in S$, an expectation of a spot price, $p \in \mathbb{R}^L_+$, or the spot price, p, in state s itself, are denoted by the pair $\omega_s := (s, p) \in S \times \mathbb{R}^L_+$. Since we are only concerned about relative prices, spot prices at $t = 1$ are restricted to the set $\mathcal{M}$.

Each agent, $i \in I$, receives an endowment, $e_i := (e_{is}) \in \mathbb{R}^{LS'}_+$, that is, a bundle of commodities, $e_{i0} \in \mathbb{R}^L_+$ at $t = 0$, and $e_{is} \in \mathbb{R}^L_+$, in each state $s \in S$, if this state prevails at $t = 1$. To harmonize notations, for every triple $(i, s, \omega) \in I \times S' \times \mathcal{M}_s$, we will also refer to $e_{i\omega} := e_{is}$. Ex post, the generic i^{th} agent's welfare is measured by a continuous utility index, $u_i : \mathbb{R}^{2L}_+ \to \mathbb{R}_+$, over her consumptions at both dates.

The financial market permits limited transfers across periods and states, via J assets, or securities, $j \in \mathcal{J} := \{1, ..., J\}$. Assets are exchanged at $t = 0$ and pay off at $t = 1$, in any contingent state, in a fixed amount of account units and/or commodities. For any forecast $\omega \in \mathcal{M}$, the cash payoffs, $v_j(\omega) \in \mathbb{R}$, of all assets, $j \in \mathcal{J}$, conditional on the occurence of (state and) price ω, define a row, $V(\omega) = (v_j(\omega)) \in \mathbb{R}^J$.

Agents can take unrestrained positions (positive, if purchased; negative, if sold), in each security, which are the components of a portfolio, $z \in \mathbb{R}^J$. Given an asset price, $q \in \mathbb{R}^J$, a portofolio, $z \in \mathbb{R}^J$, is thus a contract, which costs $q \cdot z$ units of account at $t = 0$, and promises to pay $V(\omega) \cdot z$ units tomorrow, for each expectation, $\omega \in \mathcal{M}$, if ω obtains. Similarly, we normalize first period prices, $\omega_0 := (p_0, q)$, to the set $\mathcal{M}_0$.

2.3 Information and beliefs

Ex ante, the generic agent, $i \in I$, is endowed with a private idiosyncratic set of anticipations, $P_i \subset \mathcal{M}$, according to which she believes tomorrow's true state and price (i.e., which will prevail at $t = 1$) will fall into P_i. This set may be refined from observing markets at $t = 0$. Consistently with [2], the set, $P_i \subset S \times \mathbb{R}^L$, encompasses a private information signal that the true state will be in a subset S_i of S (that is, $P_i \subset S_i \times \mathbb{R}^L$). Agents receive no wrong signal, hence, no state will prevail tomorrow, out of the pooled information set, $\underline{\mathbf{S}} := \cap_i S_i$. This yields the following definitions.

Definition 1 *A closed subset of $(S \times \mathbb{R}^L_{++}) \cap \mathcal{M}$ is called an anticipation set. Its elements are called anticipations, expectations or forecasts. We denote by $\mathcal{A}$ the set of all anticipation sets. A collection $(P_i) \in \mathcal{A}^m$ is called an anticipation structure if:*

(a) $\cap_{i=1}^m P_i \neq \varnothing$.

We denote by $\mathcal{AS}$ the set of anticipation structures. A structure, $(P'_i) \in \mathcal{AS}$, is said to refine, or to be a refinement of $(P_i) \in \mathcal{AS}$, and we denote it by $(P'_i) \leq (P_i)$, if:

(b) $P'_i \subset P_i$, $\forall i \in I$.

A refinement, $(P'_i) \in \mathcal{AS}$, of $(P_i) \in \mathcal{AS}$, is said to be self-attainable if:

(c) $\cap_{i=1}^m P'_i = \cap_{i=1}^m P_i$.

A belief is a probability, π, on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, whose support is an anticipation set, i.e., $P(\pi) \in \mathcal{A}$ (as denoted in sub-Section 2.1). A structure of beliefs is a collection of beliefs, (π_i), whose supports define an anticipation structure (i.e., $(P(\pi_i)) \in \mathcal{AS}$).

We denote by $\mathcal{B}$ and $\mathcal{SB}$, respectively, the sets of beliefs and structures of beliefs. A structure, $(\pi'_i) \in \mathcal{SB}$, is said to refine $(\pi_i) \in \mathcal{BS}$, which we denote $(\pi'_i) \leq (\pi_i)$, if $(P(\pi'_i)) \leq (P(\pi_i))$. The refinement, (π'_i), is self-attainable if $\cap_{i=1}^m P(\pi'_i) = \cap_{i=1}^m P(\pi_i)$.

Remark 1 Along the above Definition, an anticipation set is a closed set of spot prices (at $t = 1$), whose values are never zero. A belief is a probability distribution on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, which cannot put a positive weight on arbitrarily low prices. Agents' anticipations or beliefs form a structure when they have some forecasts in common. The set of common forecasts is left unchanged at a self-attainable refinement.

2.4 Consumers' behavior and the notion of equilibrium

Agents make decisions at $t = 0$, after having (possibly) inferred from markets an anticipation structure, $(P_i) \in \mathcal{AS}$, and having reached final beliefs, $(\pi_i) \in \times_{i \in I} \ \pi(P_i)$, so that, $(P(\pi_i)) = (P_i)$,[2] hereafter set as given, and referred to. The generic i^{th} agent's consumption set, $X(\pi_i)$, is that of continuous mappings from $\{0\} \cup P_i$ to $\mathbb{R}_+^L$:

$$\underline{X(\pi_i) := \ \mathcal{C} \ (\{0\} \cup P(\pi_i), \ \mathbb{R}_+^L)}.$$

A consumption, $x \in X(\pi_i)$, relates $s = 0$ to a consumption decision, $x_0 := x_{\omega_0} \in \mathbb{R}_+^L$, at $t = 0$, and, continuously on $P(\pi_i)$, every expectation, $\omega := (s, p) \in P(\pi_i)$, to a

[2] It is important to bear in mind (from Theorem 1 in [5]) that a change in equilibrium prices may result from a change in agents' beliefs, $(\pi_i) \in \times_{i \in I} \ \pi(P_i)$, but cannot bring or withdraw information to any agent.

consumption decision, $x_\omega \in \mathbb{R}^L_+$, at $t = 1$, conditional on the occurence of state s, and price p, ex post. Her preferences are represented by the V.N.M. utility function:

$$\underline{x \in X(\pi_i) \mapsto U_i(\pi_i, x) := \int_{\omega \in P(\pi_i)} u_i(x_0, x_\omega) d\pi_i(\omega).}$$

That i^{th} agent elects an optimal strategy, $(x, z) \in X(\pi_i) \times \mathbb{R}^J$, in the budget set:

$$\underline{B_i(\omega_0, \pi_i) := \{(x, z) \in X(\pi_i) \times \mathbb{R}^J : \ p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \ \ and \ \ p_s \cdot (x_\omega - e_{i\omega}) \leqslant V(\omega) \cdot z, \ \forall \omega := (s, p_s) \in P(\pi_i)\}.}$$

The above economy is denoted by $\mathcal{E}$. It retains the standard small consumer price-taker hypothesis, along which no single agent's belief, or strategy, may alone have a significant impact on prices. It is said to be standard if, moreover, it meets the following Conditions:

- ***Assumption A1***: *for each* $i \in I$, $e_i >> 0$;
- ***Assumption A2***: *for each* $i \in I$, u_i *is continuous and strictly concave*;
- ***Assumption A3***: *for any* $(i, l, t) \in I \times \mathcal{L} \times \{0, 1\}$, *the mapping* $(x_0, x_1) \mapsto \partial u_i(x_0, x_1)/\partial x^l_t$ *is defined and continuous on* $\{(x_0, x_1) \in \mathbb{R}^{2L}_+ : \ x^l_t > 0\}$, *and* $(inf_A \ \partial u_i(x_0, x_1)/\partial x^l_t) > 0$, *for every bounded subset* $A \subset \{(x_0, x_1) \in \mathbb{R}^{2L}_+ : \ x^l_t > 0\}$.

The economy's concept of equilibrium is defined as follows:

Definition 2 *A collection of prices,* $\omega_s \in \mathcal{M}_s$, *defined for each* $s \in \underline{\mathbf{S}}'$, *beliefs,* $\pi_i \in \mathcal{B}$, *and strategies,* $(x_i, z_i) \in B_i(\omega_0, \pi_i)$, *for each* $i \in I$, *is a sequential equilibrium of the economy* $\mathcal{E}$, *or correct foresight equilibrium (CFE), if the following Conditions hold:*

(a) $\forall s \in \underline{\mathbf{S}}, \ \omega_s \in \cap_{i=1}^m P(\pi_i)$;

(b) $\forall i \in I, \ (x_i, z_i) \in \arg\max_{(x,z) \in B_i(\omega_0, \pi_i)} \ U_i(\pi_i, x)$;

(c) $\forall s \in \underline{\mathbf{S}}', \ \sum_{i=1}^m (x_{i\omega_s} - e_{i\omega_s}) = 0$;

(d) $\sum_{i=1}^m z_i = 0$.

Under the above conditions, the beliefs, π_i, for each $i \in I$, or the prices, ω_s, for each $s \in \underline{\mathbf{S}}'$, are said to support the equilibrium.

2.5 No-arbitrage prices and the information they reveal

Recalling the notations of sub-Section 2.1, we first define no-arbitrage prices.

Definition 3 *Let an anticipation set, $P \in \mathcal{A}$, and a price, $q \in \mathbb{R}^J$, be given. Price q is said to be a no-arbitrage price of P, or P to be q-arbitrage-free, if:*

(a) $\nexists z \in \mathbb{R}^J : -q \cdot z \geqslant 0$ and $V(\omega) \cdot z \geqslant 0$, $\forall \omega \in P$, with one strict inequality;

We denote by $Q(P)$ the set of no-arbitrage prices of P.

Let a structure, $(P_i) \in \mathcal{AS}$, and, for each $i \in I$, the above price set, $Q(P_i)$, be given. We refer to $Q_c[(P_i)] := \cap_{i=1}^m Q(P_i)$ as the set of common no-arbitrage prices of (P_i). The structure, (P_i), is said to be arbitrage-free (respectively, q-arbitrage-free) if $Q_c[(P_i)]$ is non-empty (resp., if $q \in Q_c[(P_i)]$). We say that q is a no-arbitrage price of (P_i), and denote it by $q \in Q[(P_i)]$, if there exists a refinement, (P_i^*), of (P_i), such that $q \in Q_c[(P_i^*)]$. Moreover, if (P_i^*) is self-attainable, $q \in Q_c[(P_i^*)]$ is called self-attainable. The above definitions and notations extend to any consistent beliefs, $(\pi_i) \in \times_{i \in I} \pi(P_i)$, as denoted in sub-Section 2.1. We then refer to $Q(\pi_i) := Q(P_i)$, for each $i \in I$, and to $Q_c[(\pi_i)] := Q_c[(P_i)]$ and $Q[(\pi_i)] := Q[(P_i)]$ as, respectively, the sets of no-arbitrage prices of π_i, and of common no-arbitrage prices, and no-arbitrage prices, of the beliefs (π_i).

We notice that the symmetric refinement, (P_i^*), of any struture $(P_i) \in \mathcal{AS}$, that is, $(P_i^*) \leq (P_i)$, such that $P_j^* = \cap_{i=1}^m P_i$ for every $j \in I$, is self-attainable and arbitrage-free. Moreover, any equilibrium price on assets markets is a no-arbitrage price.

No-arbitrage prices convey information, as recalled from [5] in Claim 1.

Claim 1 *Let a structure, $(P_i) \in \mathcal{AS}$, and no-arbitrage price, $q \in Q[(P_i)]$, be given. Then, there exists a coarsest q-arbitrage free refinement of (P_i), denoted by $(\overline{\mathbf{P}}_i(q))$, in the sense that $(\overline{\mathbf{P}}_i(q))$ is q-arbitrage-free and every q-arbitrage-free refinement of (P_i) refines $(\overline{\mathbf{P}}_i(q))$. Moreover, if $q \in Q[(P_i)]$ is self-attainable, $(\overline{\mathbf{P}}_i(q))$ is self-attainable.*

Proof see [5].

Definition 4 *Given $(P_i) \in \mathcal{AS}$ and $q \in Q[(P_i)]$, the coarsest q-arbitrage-free refinement of (P_i) is said to be revealed by price q. A refinement, $(P'_i) \leq (P_i)$, is said to be price-revealable if it is the coarsest q'-arbitrage-free refinement of (P_i), for some price $q' \in Q[(P_i)]$. By extension, if $q \in Q_c[(P_i)]$, we say that (P_i) is revealed by price q.*

Given a structure, $(P_i) \in \mathcal{AS}$, a price, $q \in Q[(P_i)]$, and the refinement $(\overline{\mathbf{P}}_i(q)) \leq (P_i)$, of Claim 1, we recall the definition and property of two sequences of sets, $\{A_i^n\}_{n \in \mathbb{N}}$ and $\{P_i^n\}_{n \in \mathbb{N}}$, defined, for each $i \in I$, by induction, as follows:

- for $n = 1$, we let $A_i^1 = \varnothing$ and $P_i^1 := P_i$;
- for $n \in \mathbb{N}$ arbitrary, with A_i^n and P_i^n defined at step n, we let $A_i^{n+1} := P_i^{n+1} := \varnothing$, if $P_i^n = \varnothing$, and, otherwise,

$A_i^{n+1} := \{\overline{\omega} \in P_i^n : \exists z \in \mathbb{R}^J,\ -q \cdot z \geqslant 0,\ V(\overline{\omega}) \cdot z > 0 \textit{ and } V(\omega) \cdot z \geqslant 0,\ \forall \omega \in P_i^n\}$;

$P_i^{n+1} := P_i^n \setminus A_i^{n+1}$, i.e., the agent rules out anticipations, granting an arbitrage.

Claim 2 *Given $(P_i) \in \mathcal{AS}$ and $q \in Q[(P_i)]$, the above sequences, $\{A_i^n\}_{n \in \mathbb{N}}$ and $\{P_i^n\}_{n \in \mathbb{N}}$, satisfy the following assertion:* $\exists N \in \mathbb{N} :\ \forall n > N, \forall i \in I,\ A_i^n = \varnothing$ *and* $P_i^n = \overline{\mathbf{P}}_i(q)$.

Proof see [5].

2.6 The existence Theorem

With private idiosyncratic beliefs, a nonempty set of minimum uncertainty exists, any element of which can obtain as an equilibrium price for some beliefs today.

Definition 5 *Let Ω be the set of sequential equilibria (CFE) of the economy, $\mathcal{E}$. The minimum uncertainty set, Δ, is the subset of prices at $t = 1$, which support a CFE, namely:* $\Delta = \{\omega^* = (s^*, p^*) \in \mathcal{M} : s^* \in \underline{\mathbf{S}},\ \exists((\omega_s), (\pi_i), [(x_i, z_i)]) \in \Omega,\ \omega^* = \omega_{s^*}\}$.

The following Theorem states existence properties of a standard economy.

Theorem 1 *A standard economy, $\mathcal{E}$, its minimum uncertainty set, Δ, and an anticipation structure, $(P_i) \in \mathcal{AS}$, such that $\Delta \subset \cap_{i=1}^m P_i$ meet the following Assertions:*

(i) $\Delta \neq \varnothing$;

(ii) $\exists \varepsilon > 0 : \forall (s, p) \in \Delta,\ \forall l \in \mathcal{L},\ p^l \geqslant \varepsilon$;

(iii) a structure of beliefs, $(\pi_i) \in \Pi_{i=1}^m\ \pi(P_i)$ (along sub-Section 2.1's notations), supports a CFE if, and only if, (P_i) is arbitrage-free.

Proof see [5].

The set Δ may be seen to embed the incompressible uncertainty stemming from the fact that agents' beliefs are private. It is the set of equilibrium prices for some unknown structure of beliefs today. Along Claim 2 and Theorem 1, whenever agents are cautious enough to embed that set into their anticipation structure, (P_i), and observe a self-attainable no-arbitrage price (which always exists), $q \in \mathbb{R}^J$, they may refine their information with no price model, and reach the equilibrium anticipation structure, $(\overline{\mathbf{P}}_i(q))$. They cannot infer more information than $(\overline{\mathbf{P}}_i(q))$. Whence reached, agents may well change their beliefs, with an effect on the equilibrium price, that equilibrium price will convey exactly the same information, namely, $(\overline{\mathbf{P}}_i(q))$.

Yet, the question arises how a no-arbitrage price may obtain when (P_i) is not arbitrage-free at the outset, since agents can never agree on any assessment of assets with their in intial information, (P_i). The next Section addresses this issue.

3 A refinement path to equilibrium

3.1 Characterizing the no-arbitrage condition

We first characterize common no-arbitrage prices and structures.

Claim 3 *Let* $(P_i) \in \mathcal{AS}$, $(\pi_i) \in \times_{i \in I} \pi(P_i)$, *(using sub-Section 2.1's notations) and* $q \in \mathbb{R}^J$ *be given. The following statements are equivalent:*

(i) $q \in Q_c[(P_i)]$;

(ii) *for every* $i \in I$, *there exists a mapping,* $f : P_i \to \mathbb{R}_{++}$ *in the Riesz space* $L_2(\pi_i)$ *such that* $q = \int_{\omega \in P_i} V(\omega) f(\omega) d\pi_i(\omega)$;

Moreover, (P_i) *is arbitrage-free if and only if it meets the following AFAO Condition: There is no portfolio collection* $(z_i) \in (\mathbb{R}^J)^I$, *such that* $\sum_{i=1}^m z_i = 0$ *and* $V(\omega_i) \cdot z_i \geqslant 0$ *for every pair* $(i, \omega_i) \in I \times P_i$, *with at least one strict inequality.*

Proof Let $(P_i) \in \mathcal{AS}$ and $q \in \mathbb{R}^J$ be given.

$(ii) \Rightarrow (i)$ Assume that assertion (ii) holds and let $i \in I$ be given and $f : P_i \to \mathbb{R}_{++}$ be such that $q = \int_{\omega \in P_i} V(\omega) f(\omega) d\pi_i(\omega)$. Let $z \in \mathbb{R}^J$ be such that $-q \cdot z \geqslant 0$ and $V(\omega) \cdot z \geqslant 0$ for every $\omega \in P_i$. Assume, first, that $V(\overline{\omega}) \cdot z > 0$, for some $\overline{\omega} \in P_i$. Then, the above inequalities $V(\omega) \cdot z \geqslant 0$, which hold for every $\omega \in P_i$, and the continuity of V at $\overline{\omega}$ imply $q \cdot z = \int_{\omega \in P_i} V(\omega) \cdot z f(\omega) d\pi_i(\omega) > 0$, in contradiction with the above relation $-q \cdot z \geqslant 0$. Hence, $V(\omega) \cdot z = 0$, for all $\omega \in P_i$ and $q \cdot z = 0$, and assertion (i) holds. □

$(i) \Rightarrow (ii)$ Assume that Assertion (i) holds and let $i \in I$ and $P'_i := \{s = 0\} \cup P_i$ be given and $L_2(P'_i, \mathbb{R})$ be the set of mappings from P'_i to $\mathbb{R}$, whose restriction to P_i is in the Riesz space $L_2(\pi_i)$, endowed with the duality $(f, g) \in L_2(P'_i, \mathbb{R})^2 \mapsto < f, g > := f(0)g(0) + \int_{\omega \in P_i} f(\omega)g(\omega)d\pi_i(\omega)$, norm $f \in L_2(P'_i, \mathbb{R}) \mapsto \|f\| := \sqrt{f(0)^2 + \int_{\omega \in P_i} f(\omega)^2 d\pi_i(\omega)}$ and metric topology. Thus, $L_2(P'_i, \mathbb{R})$ is a convex metric space, with linear sub-spaces:

$$A := \{f \in L_2(P'_i, \mathbb{R}) : \exists z \in \mathbb{R}^J,\ f(0) = -q \cdot z \ and\ f(\omega) = V(\omega) \cdot z,\ \forall \omega \in P_i\};$$

$$A^{\perp} := \{f \in L_2(P'_i, \mathbb{R}) : < a, f > = 0,\ \forall a \in A\}.$$

Let $L_2(P'_i, \mathbb{R}_+)$ and $L_2(P'_i, \mathbb{R}_{++})$ be, respectively, the subsets of non-negative and strictly positive valued mappings of $L_2(P'_i, \mathbb{R})$. Assertion (i) is written $A \cap L_2(P'_i, \mathbb{R}_+) = \{0\}$. Assume, by contraposition, that $A^{\perp} \cap L_2(P'_i, \mathbb{R}_{++}) = \varnothing$, i.e., assertion (ii) fails (which implies that $\omega \in P_i \mapsto V(\omega)$ is nonzero).

From assertion (i) and above, the nonempty cone $L_2(P'_i, \mathbb{R}_{++}) - A^{\perp}$ is not dense. Hence, from ([1], Lemmas 5.44, p.188, and 5.74, p. 203) there exists a nonzero linear functional, φ, which separates $A^{\perp}$ and $L_2(P'_i, \mathbb{R}_{++})$, such that:

$$\varphi(a) = 0 \leqslant \varphi(b), \text{ for every } (a, b) \in A^{\perp} \times L_2(P'_i, \mathbb{R}_{++}).$$

From Riesz' Theorem (see [1], p. 440), there exists $f \in L_2(P'_i, \mathbb{R})$, such that $\varphi(h) = < f, h >$, for every $h \in L_2(P'_i, \mathbb{R})$. The linear space A is closed, hence, with obvious definition, $A^{\perp\perp} = A$ (see [1], p. 215). Then, from the above inequalities, the relations $f \in A^{\perp\perp} \cap L_2(P'_i, \mathbb{R}_+) \backslash \{0\} = A \cap L_2(P'_i, \mathbb{R}_+) \backslash \{0\}$ hold and contradict the above formulation, $A \cap L_2(P'_i, \mathbb{R}_+) = \{0\}$, of assertion (i). □

The fact that (P_i) meets the AFAO Condition if arbitrage-free is proved in [5]. □

Assume, now, that (P_i) meets the AFAO Condition. For each $i \in I$, we define $L_2(P'_i, \mathbb{R})$ as above and let $\mathcal{L} := \times_{i \in I} L_2(P'_i, \mathbb{R})$ be endowed with the operator, metric and topology of product spaces. We let $\mathcal{L}_+$ and $\mathcal{L}_{++}$ be the subsets of non-negative and strictly positive valued functions of $\mathcal{L}$ and A, $A^\perp$ be the linear sub-spaces:

$$A := \{(f_i) \in \mathcal{L} : (f_i(0)) = 0,\ \exists (z_i) \in \mathbb{R}^{JI} : \textstyle\sum_{i=1}^m z_i = 0,\ f_i(\omega_i) = V(\omega_i) \cdot z_i,\ \forall (i, \omega_i) \in I \times P_i\};$$

$$A^\perp := \{f \in \mathcal{L} : < a, f > = 0, \forall a \in A\}.$$

The AFAO Condition is written: $A \cap \mathcal{L}_+ = \{0\}$. If $A^\perp \cap\ \mathcal{L}_{++} = \varnothing$, the very same arguments as above apply, and, as we let the reader check, yield a contradiction. Hence, we may set as given $(f_i) \in A^\perp \cap\ \mathcal{L}_{++} \neq \varnothing$. Then, by taking $(z_i) \in (\mathbb{R}^J)^I$, such that $(z_i, z_j) = (-z_1, 0)$, for every $(i, j) \in I^2$, $i \neq 1$, $j \notin \{1, i\}$, the relation $(f_i) \in A^\perp$ yields: $\int_{\omega \in P_i} f_i(\omega) V(\omega) \cdot z d\pi_i(\omega) = \int_{\omega \in P_1} f_1(\omega) V(\omega) \cdot z d\pi_1(\omega)$, for every pair $(i, z) \in I \times \mathbb{R}^J$. Let $q := \int_{\omega \in P_1} f_1(\omega) V(\omega) d\pi_1(\omega)$. From above, $q = \int_{\omega \in P_i} f_i(\omega) V(\omega) d\pi_i(\omega)$, for every $i \in I$, and, from assertion (ii) and above, (P_i) is arbitrage-free. The proof is now complete. $\square$

3.2 The coarsest arbitrage-free refinement

We show any anticipation structure admits a coarsest arbitrage-free refinement.

Claim 4 *Any anticipation structure, $(P_i) \in \mathcal{AS}$, admits a unique coarsest arbitrage-free refinement, namely, a refinement, $(P_i^*) \leq (P_i)$, such that:*

(i) (P_i^*) *is arbitrage-free;*

(ii) *every arbitrage-free refinement of* (P_i) *is a refinement of* (P_i^*).

That coarsest arbitrage-free refinement, henceforth denoted $(\overline{\mathbf{P}}_i)$, is self-attainable.

Proof Let $(P_i) \in \mathcal{AS}$ be given and $\mathcal{R}_{(P_i)}$ be the set of arbitrage-free refinements of (P_i). That set contains the symmetric self-attainable refinement of (P_i). Let $P_i^* = \overline{\cup_{(P'_i) \in \mathcal{R}_{(P_i)}} P'_i}$, for every $i \in I$. By construction, $(P_i^*) \leq (P_i)$ is self-attainable and satisfies assertion (ii) of Claim 4. Assume, by contraposition, that (P_i^*) is not arbitrage-free, that is, from Claim 3-(iii), there exists a portfolio collection $(z_i) \in (\mathbb{R}^J)^I$, such that $\sum_{i=1}^m z_i = 0$ and $V(\omega_i) \cdot z_i \geqslant 0$ for every couple $(i, \omega_i) \in I \times P_i^*$, with at least one strict inequality, say, for $i = 1$ and $\overline{\omega} \in P_1^*$. From the continuity of $\omega \mapsto V(\omega)$, and the definition of (P_i^*), there exists $(P_i') \in \mathcal{R}_{(P_i)}$ and $\overline{\omega}_1 \in P_1'$, close enough to $\overline{\omega}$, such that, $\sum_{i=1}^m z_i = 0$, $V(\omega_i) \cdot z_i \geqslant 0$ for every couple $(i, \omega_i) \in I \times P_i'$ and $V(\overline{\omega}_1) \cdot z_1 > 0$, which (from Claim 3) contradicts the fact that (P_i') is arbitrage-free. This contradiction proves assertion (i), and completes the proof of Claim 4. □

We notice that the coarsest arbitrage-free refinement of any structure $(P_i) \in \mathcal{AS}$ is price-revelable along Defintion 4 above and coincides with (P_i) if, and only if, (P_i) is arbitrage-free.

We now examine how agents, starting from initial anticipations, $(P_i) \in \mathcal{AS}$, and endowed with no price model a la Radner, may still update their beliefs (when (P_i) is not arbitrage-free) and reach the above refinement $(\overline{\mathbf{P}}_i)$ from observing markets.

3.3 Sequential refinement through trade

Throughout, a structure, $(P_i) \in \mathcal{AS}$, is given and assumed not to be arbitrage-free. Therefore, agents cannot agree on a price assessment of assets, given this information. We study how they may narrow down in steps their expectation sets from observing exchange opportunities on financial markets. A trade-house may help reveal these exchanges, e.g., by seeking profits. We thus define, by induction on $n \in \mathbb{N}$, two sequences, $\{(A_i^n)\}_{n \in \mathbb{N}}$ and $\{(P_i^n)\}_{n \in \mathbb{N}}$, of sub-sets of $(\{\varnothing\} \cup \mathcal{M})^m$:

- we let $A_i^0 = \varnothing$ and $P_i^0 := P_i$, for each $i \in I$;
- with A_i^n and P_i^n defined at step $n \in \mathbb{N}$, for each $i \in I$, we let, for each $i' \in I$:

$$A_{i'}^{n+1} := \{\overline{\omega} \in P_{i'}^n : \exists (z_i) \in (\mathbb{R}^J)^m,\ \textstyle\sum_{i=1}^m z_i{=}0,\ V(\overline{\omega}){\cdot}z_{i'}{>}0,\ V(\omega_i){\cdot}z_i{\geqslant}0,\ \forall (i,\omega_i) \in I{\times}P_i^n\}$$
$$P_{i'}^{n+1} := P_{i'}^n \setminus A_{i'}^{n+1}$$

In the above refinement steps, agents rule out expectations, granting an arbitrage, because they may eventually trust the market over their incomplete information and realize that what they initially thought to be an arbitrage was fictitious.

Claim 5 *Let $(P_i) \in \mathcal{AS}$ be given and $(\overline{\mathbf{P}}_i)$ be its coarsest arbitrage-free refinement. Let $\{(A^n)\}_{n\in\mathbb{N}}$ and $\{(P_i^n)\}_{n\in\mathbb{N}}$, be defined from above. The following assertions hold:*
(i) $\exists N \in \mathbb{N} : \forall n > N, \forall i \in I,\ A_i^n = \varnothing$ and $P_i^n = P_i^N$;
(ii) $(P^N) = (\overline{\mathbf{P}}_i)$, along assertion (i).

Proof Let $(P_i) \in \mathcal{AS}$ be given, $\{(A_i^n)\}_{n\in\mathbb{N}}$ and $\{(P_i^n)\}_{n\in\mathbb{N}}$ be defined as above and let $P_i^* := \cap_{n\in\mathbb{N}} P_i^n = \lim_{n\to\infty} \searrow P_i^n$, for each $i \in I$.

We show, first, by induction on $n \in \mathbb{N}$, that $(\overline{\mathbf{P}}_i) \leq (P_i^n) \leq (P_i)$ for every $n \in \mathbb{N}$. The relation holds from the definition for $n = 0$, since $(P_i^0) := (P_i)$. Assume that $(\overline{\mathbf{P}}_i) \leq (P_i^n) \leq (P_i)$ holds for a given integer, $n \in \mathbb{N}$. Then, for each $i \in I$, P_i^n is closed, and so is P_i^{n+1} from the definition and the continuity of $\omega \mapsto V(\omega)$. Assume, by contraposition, that, for some $n \in \mathbb{N}$, and some $i \in I$, say $i = 1$, $\overline{\mathbf{P}}_1 \subset P_1^n$ and $\overline{\mathbf{P}}_1 \not\subseteq P_1^{n+1}$. Then, there exist $\overline{\omega} \in \overline{\mathbf{P}}_1 \cap A_1^{n+1}$ and $(z_i) \in (\mathbb{R}^J)^m$, such that $\sum_{i'=1}^m z_i = 0$, $V(\overline{\omega}) \cdot z_1 > 0$ and $V(\omega_i) \cdot z_i \geqslant 0$, for every $(i, \omega_i) \in I \times \overline{\mathbf{P}}_i \subset I \times P_i^n$, which contradicts Claims 3 and 4, along which $(\overline{\mathbf{P}}_i)$ is arbitrage-free and meets the AFAO Condition.

Hence, the relations $(\overline{\mathbf{P}}_i) \leq (P_i^n) \leq (P_i)$ hold for all $n \in \mathbb{N}$, which implies, passing to the limits on nonempty intersections of compact sets: $(\overline{\mathbf{P}}_i) \leq (P_i^*) \leq (P_i)$.

For each $i \in I$, let $Z_i^{on} := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0,\ \forall \omega \in P_i^n\}$. Since $\{(P_i^n)\}_{n\in\mathbb{N}}$ is non-increasing, the sequence of vector spaces, $\{\times_{i\in I} Z_i^{on}\}$, is non-decreasing in $(\mathbb{R}^J)^m$, hence, stationary. We let $N \in \mathbb{N}$ be such that $\times_{i\in I} Z_i^{on} = \times_{i\in I} Z_i^{oN}$, for every $n \geqslant N$. Assume, by contraposition, that assertion (i) of Claim 5 fails, that is:

$$\forall n \in \mathbb{N}, \exists (\omega_{i_n}^n, (z_i^n)) \in P_{i_n}^n \times \mathbb{R}^{Jm} : \ \sum_{i=1}^m z_i^n = 0,\ V(\omega_{i_n}^n) \cdot z_{i_n}^n > 0 \ and \ V(\omega_i) \cdot z_i^n \geqslant 0, \forall (i, \omega_i) \in I \times P_i^n.$$

From the definition of (P_i^n) and (P_i^{n+1}), the above portfolios satisfy, for all $n \in \mathbb{N}$, $(z_i^n) \notin \times_{i\in I} Z_i^{on}$ and $(z_i^n) \in \times_{i\in I} Z_i^{o(n+1)}$, which is impossible, from above, if $n \geqslant N$. This contradiction proves that assertion (i) of Claim 5 holds, for the integer $N \in \mathbb{N}$ introduced above. Moreover, $(P_i^*) = (P_i^N)$, is q-arbitrage-free (since $A_i^{N+1} = \varnothing$, for each $i \in I$), which yields, from Claim 4 and above: $(P_i^*) \leq (\overline{\mathbf{P}}_i) \leq (P_i^*) \leq (P_i)$. That is, $(\overline{\mathbf{P}}_i) = (P_i^*) = (P_i^N)$, and assertion (ii) of Claim 5 holds. This completes the proof. $\square$

Thus, agents may always refine their information with no price model (and even no market price to observe) and reach an arbitrage-free anticipation structure, and an equilibrium along Theorem 1, if they are cautious enough to embed the set Δ into their anticipations. We suggested in [5] the inference of Δ or of a bigger set might result from past price observation.

Once agents have reached the coarsest arbitrage-free refinement, $(\overline{\mathbf{P}}_i)$, from observing trade opportunities, they have no means of changing their anticipations. All equilibrium prices, which belong to $Q_c[(\overline{\mathbf{P}}_i)]$, reveal the coarse structure, $(\overline{\mathbf{P}}_i)$. Along Theorem 1, beliefs may well change and lead equilibrium prices to change, this will not modify any agent's anticipation set. In that sense, the path to equilibrium discards rational expectations.

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